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BOUNDARY VALUE AND EXPANSION PROBLEMS: ALGEBRAIC BASIS OF THE THEORY.*

By R. D. Carmichael.

1. *Introduction.*—Linear problems in various forms have been central in the development of extensive chapters of mathematical analysis. Many of the profound phenomena of nature are subject to laws whose expression in mathematical form gives rise to fundamental linear problems of several kinds. Logically the simplest and historically the first to be treated in detail of the linear problems of pure mathematics are those having to do with systems of linear algebraic equations. And these hold the place of greatest importance both on account of their simplicity and complete development and on account of their suggestiveness in other linear problems. In fact it is true that a wide and important range of transcendental linear problems emerge from a direct consideration of the natural limiting cases of algebraic systems under the guidance of current problems of transcendental analysis.

The deep-lying connection between algebraic and transcendental problems of linear character has often afforded a useful help in the investigation of the latter, particularly in the case of integral equations. The object of the present investigation (of which the first portion is here presented) is to treat a considerable range of transcendental boundary value and expansion problems in close connection with certain algebraic problems of which they are the limiting cases.

It is necessary first of all to formulate and solve the algebraic problems with special reference to their use as a heuristic guide in discovering truths and their proofs for the limiting cases which are matters of prime interest. From the outset classic theorems for certain transcendental cases throw a light back upon the algebraic questions and are of great value in formulating and in solving the algebraic problems. Thus we have from the outset a valuable interaction between the algebraic problems on the one hand and the transcendental problems on the other. The great intimacy of this interaction becomes increasingly apparent as the investigation proceeds.

In his researches on integral equations, integro-differential equations, functions of lines, and permutable functions, Volterra† has made frequent and consistent use of the guide to the transcendental problems afforded by looking upon them as limiting cases of algebraic problems. This seems to be the first instance of the systematic and profound use of the principle of guidance which in a modified or extended form is made to lead the way in our study of boundary value and expansion problems. The same general principle served Fredholm* as a heuristic guide in his fundamental memoir on integral equations; and Hilbert† carried through the limiting processes which served Fredholm merely as a guide and deduced his results as limiting cases of algebraic propositions when the number of variables becomes infinite.

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* Presented to the American Mathematical Society, April, 1920.

† See his "Le cours sur les équations intégrales et les équations intégro-différentielles," 1913, where an exposition of his work is given with references to his earlier memoirs.

Though the guide to transcendental problems afforded by the properties of algebraic equations has attracted the greatest attention in the field of integral equations and the closely related matters, it was not here that the method was first employed heuristically. It is indeed the unpublished method which Sturm employed in his fundamental researches on differential equations of the second order. The differential equation with boundary conditions may be looked upon as the limiting case of a corresponding difference equation with boundary conditions, the latter being but an abbreviated form of a certain restricted system of algebraic equations. The passage to the limit from the difference equation to the differential equation may be carried through rigorously in the case of several boundary value problems, as was shown by Porter in 1902 (more than two years before Hilbert took a similar step for integral equations).‡

An examination of the previous study of differential equations from the point of view in consideration brings out the fact that a given differential equation with boundary conditions may be realized in an infinite number of ways as a limiting form of algebraic systems. The investigator must therefore exercise care in choosing that algebraic system of which a maximum number of properties persist if he is to obtain the greatest advantage from the interaction of the two problems. The results in § 6 of the present paper were suggested by classic theorems for differential equations; success in arriving at them required first the choice of a particular form of algebraic system to yield the differential equation and the generalization of this form so as to maintain the requisite characteristics. The scaffolding by which the results were discovered has not been reproduced in the section.

Similarly, one may look upon partial differential equations, integro-differential equations, and various types of mixed equations, each as a limiting form of appropriate algebraic systems. Moreover, one may consider sets of algebraic systems and their limiting forms of such sort as to furnish guidance in the study of systems consisting of differential equations, difference equations, integral equations, and so forth, and any one of many combinations of these. The algebraic theory developed in the present paper is of such sort as to furnish guidance in the investigation of boundary value and expansion problems associated with each of these various types of equations and systems, as I shall make clear in later papers recording the results of the present general investigation.

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* *Acta Mathematica,* 27 (1903): 365.

† *Göttingen Nachrichten,* 1904, p. 49.

‡ For references and a fuller discussion of matters referred to in this paragraph the reader may consult Bôcher's lecture before the Fifth International Congress of Mathematicians and his "Les méthodes de Sturm."

The matter developed in this paper may be outlined as follows: In § 2 are recorded preliminary considerations relative to sets of adjoint algebraic systems involving a finite number of parameters subject to determination so as to render the systems consistent. The conjugate character of the solutions of these systems is treated in § 3 and properties are established similar to those of orthogonality and biorthogonality in the theory of functions. These results are applied in § 4 in deriving the 'expansions' of sets of constants in terms of conjugate sets. A certain graphic representation of solutions is defined and analyzed in § 5 and is employed in § 6 in deriving and extending certain theorems analogous to the Sturmian zero separation theorems for linear differential equations of the second order. In § 7 there is a discussion of the general character of a fundamental limiting operation (which is to be employed frequently in analyzing the transcendental problems which are the goal of the investigation); and in § 8 a second type of limiting case is briefly treated. Variation of parameters and Green's functions for algebraic systems are the topics discussed in § 9. Finally, in § 10, a method is given for the condensation of the algebraic expansions of § 4 into contour integrals, the methods and the results being analogous to those which Birkhoff (see reference in § 10) has developed for a certain class of differential systems.

2. *Adjoint Algebraic Systems.*—Among the natural ways of introducing the so-called adjoint algebraic system in connection with a given algebraic system is one which is intimately related to the usual method of introducing the adjoint differential expression in connection with a given differential expression. We recall that the differential expression $M(v)$ which is adjoint to a given homogeneous linear differential expression $L(u)$ is gotten by seeking a v such that an indefinite integral of $vL(u)$ can be written as a homogeneous linear differential expression in u of order one lower than that of $L(u)$. Following up the analogy and employing a sum as to i in place of the integral used in the case of differential expressions, we are led to seek the conditions on $y_1, y_2, \cdots, y_n$ such that for given α_{ij} the coefficients

β_{ij} exist of such sort that we have

$$\sum_{i=1}^{n} y_i \sum_{j=1}^{n} \alpha_{ij} x_j \equiv \sum_{\substack{j=1 \\ j \neq k}}^{n} \beta_{kj} x_j$$

for each k of the set $1, 2, \cdots, n$. We see that we must have

$$\sum_{i=1}^{n} \alpha_{ik} y_i = 0$$

for every k. Thus with the first of the two sets of linear forms

$$\sum_{j=1}^{n} \alpha_{ij} x_j, \qquad \sum_{j=1}^{n} \alpha_{ji} y_j, \qquad i = 1, 2, \cdots, n,$$

we are led to associate the second. If we start with the latter it is clear that we shall be led in the same way to the former. The two systems of forms are said to be adjoint each to the other.

As an analogue of the classic Lagrange identity [involving $vL(u) - uM(v)$] in the theory of adjoint differential expressions, we have the readily verified identity

$$\sum_{i=1}^{n} \left\{ y_i \sum_{j=1}^{n} \alpha_{ij} x_j - x_i \sum_{j=1}^{n} \alpha_{ji} y_j \right\} \equiv 0,$$

which will be found to serve us in a fundamental way as a guide to the method of approach to numerous problems.

Let us now consider the r homogeneous linear algebraic systems in the unknown quantities x,

$$(1) \quad \sum_{j=1}^{n_h} (a_{0hij} + \lambda_1 a_{1hij} + \lambda_2 a_{2hij} + \cdots + \lambda_r a_{rhij}) x_{hj} = 0, \qquad i = 1, 2, \cdots, n_h,$$

a separate system being formed for each value h of the set $1, 2, \cdots, r$. Here $\lambda_1, \lambda_2, \cdots, \lambda_r$ are r parameters by the choice of which the simultaneous consistency of the r systems is to be secured. With these systems let us associate the adjoint systems

$$(2) \quad \sum_{j=1}^{n_h} (a_{0hji} + \lambda_1 a_{1hji} + \lambda_2 a_{2hji} + \cdots + \lambda_r a_{rhji}) y_{hj} = 0, \quad i = 1, 2, \cdots, n_h.$$

A necessary and sufficient condition that each of the r systems in (1) shall have solutions not identically zero is that each of the equations

$$(3) \qquad |a_{0hij} + \lambda_1 a_{1hij} + \cdots + \lambda_r a_{rhij}| = 0, \qquad h = 1, 2, \cdots, r,$$

shall be satisfied, where for a given value of h the first number is a determinant of order n_h whose element in ith row and jth column is that which is written out explicitly. Obviously we have the same necessary and sufficient condition that each of the r systems in (2) shall have solutions not identically zero.

A set of (finite) values of $\lambda_1, \lambda_2, \cdots, \lambda_r$ for which (1) [(2)] has for each h a solution not identically zero will be called *a set of characteristic values for* (1) [(2)]. We have just seen that (1) and (2) have the same sets of characteristic values, since the sets of characteristic values are the (finite) solutions of system (3) considered as a system of equations for determining $\lambda_1, \lambda_2, \cdots, \lambda_r$.

We shall suppose that the coefficients a are of such sort that each of the expanded determinants in (3) has explicitly present at least one of the parameters λ. Then system (3) is a set of algebraic equations r in number and involving r unknown quantities. We assume furthermore that the coefficients a are such that these r relations are independent in such wise that we have only a finite number of sets of characteristic values.

These sets of characteristic values we shall then denote by

$$\lambda_1^{(\rho)}, \lambda_2^{(\rho)}, \cdots, \lambda_r^{(\rho)}$$

for varying values of ρ, the two ordered sets being distinct for two distinct values of ρ. The corresponding solutions of (1) and (2) we shall then denote by

$$(4) \qquad x_{hj}^{(\rho)}, \; y_{hj}^{(\rho)}, \; j = 1, 2, \cdots, n_h, \; h = 1, 2, \cdots, r.$$

3. *Conjugate Character of the Solutions of* (1) *and* (2). With the notation adopted in the preceding section for solutions of (1) and (2) we have relations which may be written in the form

$$(5) \quad \sum_{j=1}^{n_h} (a_{0hij} + \lambda_1^{(\rho)} a_{1hij} + \cdots + \lambda_r^{(\rho)} a_{rhij}) x_{hj}^{(\rho)} = 0, \qquad i = 1, 2, \cdots, n_h,$$

$$(6) \quad \sum_{j=1}^{n_h} (a_{0hji} + \lambda_1^{(\rho)} a_{1hji} + \cdots + \lambda_r^{(\rho)} a_{rhji}) y_{hj}^{(\sigma)} + \sum_{j=1}^{n_h} \sum_{s=1}^{r} (\lambda_s^{(\sigma)} - \lambda_s^{(\rho)}) a_{shji} y_{hj}^{(\sigma)} = 0, \qquad i = 1, 2, \cdots, n_h,$$

systems being formed for each value h of the set $1, 2, \cdots, r$. For fixed h and i let us multiply the former of these equations member by member by $y_{hi}^{(\sigma)}$ and the latter by $-x_{hi}^{(\rho)}$, and let us then add the two resulting equations member by member. Still holding h fixed in the relation so obtained, sum as to i from 1 to n_h. That part of the first member of the resulting equation which comes from the first member of (5) and the first line of the first member of (6) balances to zero and we have a relation which may readily be reduced to the form

$$(7) \qquad \sum_{s=1}^{r} (\lambda_s^{(\sigma)} - \lambda_s^{(\rho)}) \sum_{i=1}^{n_h} \sum_{j=1}^{n_h} a_{shji} x_{hi}^{(\rho)} y_{hj}^{(\sigma)} = 0, \qquad h = 1, 2, \cdots, r.$$

Here we have r equations involving the r quantities $\lambda_s^{(\sigma)} - \lambda_s^{(\rho)}$, $s = 1, 2, \cdots, r$, these being not simultaneously zero when ρ and σ are different. Hence the determinant of the coefficients of these quantities in (7) must have the value zero. If we employ different summation variables i and j for the different rows in this determinant, the relation may be put in the form

$$\left| \sum_{i_h=1}^{n_h} \sum_{j_h=1}^{n_h} a_{shj_hi_h} x_{hi_h}^{(\rho)} y_{hj_h}^{(\sigma)} \right| = 0, \qquad \rho \neq \sigma,$$

where the element written explicitly is that in the sth column and the hth row. It is easy to see that this relation may be put into the following more convenient form:

$$\text{(8)} \qquad \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{i_1j_1\dots i_rj_r} \prod_{h=1}^{r} x_{hi_h}^{(\rho)} y_{hj_h}^{(\sigma)} = 0, \qquad \rho \neq \sigma,$$

where $D_{i_1j_1\dots i_rj_r}$ denotes the determinant

$$D_{i_1j_1\cdots i_rj_r} = \begin{vmatrix} a_{11j_1i_1} & a_{21j_1i_1} & \cdots & a_{r1j_1i_1} \\ a_{12j_2i_2} & a_{22j_2i_2} & \cdots & a_{r2j_2i^{2}} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1rj_ri_r} & a_{2rj_ri_r} & \cdots & a_{rrj_ri_r} \end{vmatrix}.$$

This relation for varying ρ and σ, expresses the fundamental conjugate character of the solutions of (1) and (2) with respect to each other.

In (3) let the $\lambda_1, \cdots, \lambda_r$ be replaced by the set of characteristic values $\lambda_1^{(\rho)}, \cdots, \lambda_r^{(\rho)}$. For a given h let $\alpha_{hij}^{(\rho)}$ denote the cofactor of the element in the ith row and jth column of the determinant in the resulting first member. Then, if for each h some one of these cofactors is different from zero, we have a relation of the form

$$x_{h1}^{(\rho)} : x_{h2}^{(\rho)} : \cdots : x_{hn_h}^{(\rho)} = \alpha_{hi1}^{(\rho)} : \alpha_{hi2}^{(\rho)} : \cdots : \alpha_{hin_h}^{(\rho)}.$$

Similarly, we have

$$y_{h1}^{(\rho)} : y_{h2}^{(\rho)} : \cdots : y_{hn_h}^{(\rho)} = \alpha_{h1j}^{(\rho)} : \alpha_{h2j}^{(\rho)} : \cdots : \alpha_{hn_hj}^{(\rho)}.$$

Employing ξ_h and η_h as the (necessarily non-zero) factors of proportionality, we have

$$x_{hk}^{(\rho)} = \xi_k \alpha_{hik}^{(\rho)}, \qquad y_{hk}^{(\rho)} = \eta_h \alpha_{hkj}^{(\rho)}.$$

If we denote the first member of (8) by $S(\rho, \sigma)$ we have then

$$\text{(9)} \qquad \frac{S(\rho, \rho)}{\prod_{h=1}^{r} \xi_h \eta_h} = \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{i_1j_1\dots i_rj_r} \prod_{h=1}^{r} \alpha_{hii_h}^{(\rho)} \alpha_{hj_hj}^{(\rho)}.$$

Now the second member of (4) depends only on the coefficients in equations (1) and (2). If i and j exist so that this second member is different from zero we have $S(\rho, \rho) \neq 0$.

Since we shall be interested in these algebraic results primarily for their use as a heuristic guide in certain transcendental problems we shall naturally exclude from consideration such exceptional cases as are complicated or otherwise fail to serve our purpose. Consequently we shall suppose now and usually henceforth that *the coefficients of* (1) *and* (2) *are of such character that the second member of* (9) *is different from zero for every* ρ. Then we have the following result:

The first member of (8) *is equal to zero when* $\rho \neq \sigma$ *and is different from zero when* $\rho = \sigma$.

In the special case in which (1) is self-adjoint and the x's and the y's, for a given ρ, are taken to be the same, it is clear that the condition that all determinants in (8) shall have the same sign (excluding therefore zero values) is a sufficient condition for the non-vanishing of the first member of (8) when $\rho = \sigma$.

In order to bring out clearly the nature of the conditions of conjugacy which we have just established let us consider the special case in which $r = 1$. Systems (1) and (2) may then be conveniently written in the form

$$(10) \qquad \sum_{j=1}^{n} (c_{ij} + \lambda b_{ij}) x_j = 0, \qquad \sum_{j=1}^{n} (c_{ji} + \lambda b_{ji}) y_j = 0, \qquad i = 1, 2, \cdots, n.$$

The conditions of conjugacy reduce to

$$(11) \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} x_i^{(\rho)} y_j^{(\sigma)} \begin{cases} = 0 \text{ if } \sigma \neq \rho, \\ \neq 0 \text{ if } \sigma = \rho. \end{cases}$$

If we have the more special case in which $b_{ii} = b_i$, $b_{ij} = 0$ if $j \neq i$, these conditions become

$$(12) \qquad \sum_{i=1}^{n} b_i x_i^{(\rho)} y_i^{(\sigma)} \begin{cases} = 0 \text{ if } \sigma \neq \rho, \\ \neq 0 \text{ if } \sigma = \rho. \end{cases}$$

When $b_i = 1$ for every i these become merely the usual conditions of biorthogonality; and these in turn reduce to the usual conditions of orthogonality in case the two systems in (10) are identical and the same solutions x and y are taken for the two systems. From this it is seen that our formulæ give extensive generalizations of classic elementary relations of wide usefulness.

Let us consider a special case of equations (1) and (2) giving a rather simple generalization of the results in the preceding paragraph. Let us suppose that each system in (1) contains but a single parameter, say that λ_h is the only parameter in the system for a given value of h. Then the r

equations (3) involve each a single one of the r parameters $\lambda_1, \cdots, \lambda_r$, so that the possible values of any given parameter are obtained by solving a single equation explicitly given. All possible combinations of solutions are allowable in forming the various sets of characteristic values. The determinants in (8) reduce in the present case to their diagonal elements, so that we have for this case the simpler conditions of conjugacy

$$\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r}\left(\prod_{h=1}^{r} a_{hhj_hi_h}x_{hi_h}^{(\rho)}y_{hj_h}^{(\sigma)}\right)\begin{cases} = 0 \text{ if } \sigma \neq \rho, \\ \neq 0 \text{ if } \sigma = \rho. \end{cases} \tag{13}$$

If for a given subscript k no characteristic value of λ_k is equal to zero we may divide systems (1) and (2) by λ_k and so introduce a new set of r parameters $1/\lambda_k$, λ_i/λ_k for $i \neq k$. These enter linearly and in the same way as the original parameters. It is easy to see that the solutions x and y are the same as before. Consequently we have new sets of relations similar to (8) and (13) and differing from them only in having a_{khij} replaced throughout by a_{0hij}.

4. *Expansions by Use of Conjugate Sets.*—Let m be the number of sets of characteristic values of $\lambda_1, \cdots, \lambda_r$ and let us suppose that the constants $z_{i_1i_2\cdots i_r}$ for i_h varying from 1 to n_h for each h may be "expanded" in the form

$$z_{i_1i_2\cdots i_r} = \sum_{k=1}^{m} c_k x_{1i_1}^{(k)} x_{2i_2}^{(k)} \cdots x_{ri_r}^{(k)}, \tag{14}$$

where the c_k are independent of the subscripts i_h. Then the properties of conjugacy developed in § 3 are available for an immediate determination of the values of the c_k—much after the manner of determining the coefficients of a Fourier expansion. For this purpose we multiply both sides of equation (14) by $y_{1j_1}^{(l)} \cdots y_{rj_r}^{(l)}$ times the determinant $D_{i_1j_1\cdots i_rj_r}$ explicitly written in connection with (8) and in the resulting equation sum as to the i and the j. In view of the conditions of conjugacy the resulting quantity in the second member depends on only one of the constants c_k, namely c_l; and we then have

$$c_l = \frac{\displaystyle\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r}\left(D_{i_1j_1\cdots i_rj_r} z_{i_1i_2\cdots i_r}\prod_{h=1}^{r} y_{hj_h}^{(l)}\right)}{\displaystyle\sum_{i_1=1}^{n_1}\sum_{j_1=1}^{n_1}\cdots\sum_{i_r=1}^{n_r}\sum_{j_r=1}^{n_r}\left(D_{i_1j_1\cdots i_rj_r}\prod_{h=1}^{r} x_{hi_h}^{(l)}y_{hj_h}^{(l)}\right)}. \tag{15}$$

It is easy to obtain broad sufficient conditions for the validity of the "expansion" (14) for any given set $z_{i_1i_2\cdots i_r}$ of $n_1n_2 \cdots n_r$ constants. We suppose that $m = n_1n_2 \cdots n_r$ so that the number of distinct solutions of (3) is equal to the product of the degrees of the several equations in that system. If then we look upon (14) as a system of m equations for determining the quantities $c_1, c_2, \cdots, c_m$ it is sufficient that the determinant D of this system shall be different from zero. But we have seen (in § 3) that $x_{ht}^{(k)} = \xi_h \alpha_{hit}^{(k)}$ for some appropriate i (dependent perhaps upon k), the quantities ξ_h being different from zero. Hence the determinant D will be different from zero if the corresponding determinant formed with $\alpha_{hit}^{(k)}$ in place of $x_{ht}^{(k)}$ be different from zero. This last is a condition on the coefficients a in equation (1). Thus we may express in terms of the coefficients in (1) broad general conditions which are sufficient to ensure the validity of the expansion (14), where the coefficients c_l have the values given in (15).

It is not in accordance with our present plan to seek those modifications of the foregoing expansion which are necessary in the exceptional cases excluded from the preceding discussion. It does not appear that we shall need the theory for these cases in order to avail ourselves of the heuristic guide afforded by the algebraic theory in its non-exceptional and more general aspects.

In order to bring out clearly the essential simplicity of these formulæ let us observe that for the special cases involved in equations (11) and (12) we have

$$(16) \qquad z_i = \sum_{k=1}^{n} c_k x_i^{(k)}, \qquad i = 1, 2, \cdots, n,$$

where c_k has one or the other of the values

$$(17) \qquad c_k = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} z_i y_j^{(k)}}{\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} x_i^{(k)} y_j^{(k)}}, \qquad \frac{\sum_{i=1}^{n} b_i z_i y_i^{(k)}}{\sum_{i=1}^{n} b_i x_i^{(k)} y_i^{(k)}},$$

according as we have the case of (11) or of (12). If $b_i = 1$ the latter reaches the maximum of elegance, at least if equations (10) are then self-adjoint and the same solutions x and y of the two systems are taken.

For the general case when $m = n_1 n_2 \cdots n_r$ there is a certain gain in suggestiveness if we denote the parameters and solutions of (1) and (2) by

$$\lambda_h^{(k_1 k_2 \ldots k_r)}, \quad x_{hi_h}^{(k_1 k_2 \ldots k_r)}, \quad y_{hj_h}^{(k_1 k_2 \ldots k_r)},$$

where k_s runs over the set $1, 2, \cdots, n_s$. Then (14) takes the form

$$(18) \qquad z_{i_1 i_2 \ldots i_r} = \sum_{k_1=1}^{n_1} \cdots \sum_{k_r=1}^{n_r} \left(c_{k_1 k_2 \ldots k_r} \prod_{h=1}^{r} x_{hi_h}^{(k_1 k_2 \ldots k_r)} \right).$$

The coefficients c are determined in the same manner as before. Just as we may look upon (16) and (17) as the analogue of expansions of functions of one variable in orthogonal functions of that variable (including the special case of Fourier series) so shall we be able to look upon (18) as the analogue of like expansions (including double Fourier series) by which functions of several variables are expressed in terms of products of functions each of one variable—such expansions being of the kind used by the phsyicists in the solution of many classical problems. Indeed these are much more than analogues; transcendental expansion problems of several well-known types (including those mentioned) and of many new types are as a matter of fact limiting cases of the algebraic expansion problem which we are treating (and we are developing the latter primarily for its applications to the former).

In the foregoing expansion formulæ the number of subscripts on the "function" z to be expanded in terms of a given set of constants by aid of the property of conjugacy is equal to the number of parameters λ involved in the original system of algebraic equations. When we proceed to our transcendental limiting cases we shall see that these algebraic results are in the form best suited to applications to equations involving functions of one variable, as ordinary differential equations, ordinary difference equations, integral equations, and certain types of mixed systems. In order to obtain a form suitable as the heuristic guide in problems involving partial differential equations or equations of any type in which we have functions of more than one variable in the original equations, it is desirable to look upon our expansion formulæ in a way slightly different from that in evidence in the foregoing work.

We can best bring out what is needful by considering first the particular expansion of equation (16). Let us suppose that n is the product $\mu\nu$ of two integers. Let us replace the subscript i, running over the set $1, 2, \cdots, n$, by the double subscript ij where i runs over $1, 2, \cdots, \mu$, and j over $1, 2, \cdots, \nu$. Then equations (16) become

$$(19) \qquad z_{ij} = \sum_{k=1}^{\mu\nu} c_k x_{ij}^{(k)}, \qquad i = 1, 2, \cdots, \mu, \qquad j = 1, 2, \cdots, \nu.$$

The coefficients c_k are of course determined in the same way as before but are naturally expressed in a form differing from that in (17) by having the single summation as to i replaced by a double summation as to i and j.

5. *Graphic Representations of Solutions.*—Let us consider the graphical representation of the set $u_1, u_2, \cdots, u_n$ of constants in the following manner. On any given straight line segment (as for instance the points s such that $a \leqq s \leqq b$) let us erect n perpendiculars two of which are at the ends of the segment while the other $n - 2$ are evenly or evenly distributed on the interior of the interval. Let these be marked from left to right by the numbers $1, 2, \cdots, n$; and consider them as analogous to the n coördinate axes of a space of n dimensions. Considering $(u_1, u_2, \cdots, u_n)$ as a point in space of n dimensions we may now represent this single point by a certain

junction of n points in our figure, one on each of the axes, the ith one being on the ith axis and at a distance $|u_i|$ from the original segment and above it or below it according as u_i is positive or negative. Now join by straight line segments the point on each interior axis to the points on the two adjacent axes. We thus obtain a broken line. We shall say that this broken line is the graphic representation* of the point $(u_1 u_2 \cdots u_n)$ in n dimensions or of the set of constants $u_1, u_2, \cdots, u_n$. The points at which (values of s for which) this broken line cuts the original line segment (viewed as the axis of s) we shall call the *zeros* of the set u_i, in analogy with the usual terminology for the zeros of a function $u(s)$.

Now let us suppose that the same set of axes is used for the representation of the n sets $x_i^{(k)}$, $k = 1, 2, \cdots, n$, and the set z_i of equation (16). Let $x^{(k)}(s)$, $k = 1, 2, \cdots, n$, and $z(s)$ denote the functions of the continuous variable s represented by the broken lines corresponding as indicated above to the foregoing sets of constants. Then we shall show that

$$z(s) = \sum_{k=1}^{n} c_k x^{(k)}(s), \tag{20}$$

where the constants c_k have the same values as in (16). For this purpose it is sufficient to consider the graph between two consecutive vertical axes, say between the ith and the $(i+1)$th. Let s_i and s_{i+1} be the values of s at the intersections of these vertical axes with the s-axis. Then between the two vertical axes in consideration we have

$$z(s) = z_i + \frac{s - s_i}{s_{i+1} - s_i}(z_{i+1} - z_i), \qquad x^{(k)}(s) = x_i^{(k)} + \frac{s - s_i}{s_{i+1} - s_i}(x_{i+1}^{(k)} - x_i^{(k)}),$$
$$k = 1, 2, \cdots, n.$$

From these relations and from (16) with the subscripts i and $i + 1$ it follows that (20) is valid for the interval in consideration. Hence (20) is valid for all values of s on the range $a \leqq s \leqq b$ of the original line segment.

Let us consider the like matter for functions of two subscripts and expansions of the form (19). For the representation of u_{ij} for $i = 1, 2, \cdots, \mu$, and $j = 1, 2, \cdots, \nu$, we shall start from ν planes, one for each value of j. In each of these we use for s the same range $a \leqq s \leqq b$ and place the planes in order $1, 2, \cdots, \nu$ for j each directly in front of the preceding one and arrange the vertical axes so that the axes in any one plane are the orthogonal projections on that plane of the axes in any other plane. Then for fixed j we extend the u_{ij} to $u_j\ (s)$ by linear interpolation as before. Then for each value of s we connect the points $u_j(s)$, $j = 1, 2, \cdots, \nu$, by straight

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[*] Essentially the same graphic representation has been employed by M. B. Porter in *Annals of Mathematics* (2), 3 (1901), p. 56.

line segments joining the consecutive points. We thus get a sort of broken surface affording the graphical representation of a function $u(s, t)$, gotten (we may say) by linear interpolation from the points u_{ij}. We call this the graphical representation of the set of constants u_{ij} for $i = 1, 2, \cdots, \mu$, and $j = 1, 2, \cdots, \nu$. Let $c \leqq t \leqq d$ define the range of t in this representation, this range evidently depending on the positions of the ν planes employed in setting up the graphical representation.

Let us replace the subscripts i, j in the functions in (19) by the continuous variables s, t in accordance with the method just indicated. It is easy to prove that we then have

$$z(s, t) = \sum_{k=1}^{\mu\nu} c_k x^{(k)}(s, t), \qquad a \leqq s \leqq b, \qquad c \leqq t \leqq d, \tag{21}$$

where the constants c_k have the same values as in equation (19). In fact the argument associated with (20) shows that we have

$$z_j(s) = \sum_{k=1}^{\mu\nu} c_k x_j^{(k)}(s), \qquad j = 1, 2, \cdots, \nu.$$

A repetition of the same argument for these last equations with s fixed leads at once to relation (21).

Similar extensions may be associated with equation (14) so that we readily come through to the relation

$$z(s_1, s_2, \cdots, s_r) = \sum_{k=1}^{m} c_k x_1^{(k)}(s_1) x_2^{(k)}(s_2) \cdots x_r^{(k)}(s_r),$$

where the c_k have the same values as in (14) and the functions involved depend on continuous variables. Further generalizations may also be made in the direction suggested by the last paragraph of § 4.

6. *On the Distribution of Zeros of Solutions.*—The graphic representation of solutions described in the preceding section we shall find to afford one of the most useful ideas in the heuristic guide to transcendental problems arising from the algebraic theory which we are now developing. In order to have at hand an aspect of its usefulness not brought out directly by the applications to expansion problems, let us consider a certain fundamental question concerning the relative distribution of zeros of the solutions of certain types of algebraic systems.

Let us first consider a system of n homogeneous linear algebraic equations

$$\sum_{j=1}^{n+2} a_{ij} x_j = 0, \qquad i = 1, 2, \cdots, n, \tag{22}$$

in the $n + 2$ unknown quantities $x_1, x_2, \cdots, x_{n+2}$; and let us suppose that the matrix of this system is of rank n. The system has two solutions $x_i = u_i$

and $x_i = v_i$ such that every solution is expressible in the form $x_i = cu_i + dv_i$, where c and d are constants, while conversely every such x_i is a solution. The sets u_i and v_i are linearly independent. If $x_i = u_i$ and $x_i = v_i$ are any two linearly independent solutions and if Δ_{kl} denotes the determinant of the matrix obtained from the matrix of coefficients in (22) by striking out the kth and lth columns, it may be shown that Δ_{kl} *and the determinant* $u_kv_l - u_lv_k$ *are both zero or neither zero, provided that when* $\Delta_{kl} = 0$ *we do not have* $\Delta_{ml} = 0 = \Delta_{km}$ *for every* m *of the set* $1, 2, \cdots, n - 2$, *except* k *and* l.

In order to prove the last statement, let us write equations (22) in the form

$$\sum_{\substack{j=1 \\ j\neq k,\, l}}^{n+2} a_{ij}x_j = - a_{ik}x_k - a_{il}x_l.$$

Then we have

$$\Delta_{kl}x_m = \pm \Delta_{ml}x_k \pm \Delta_{km}x_l, \tag{23}$$

the ambiguous signs $\pm$ being obviously determinate.

Let us suppose that $\Delta_{kl} \neq 0$. Then we can not have simultaneously $u_k = 0$, $u_l = 0$; for, if so, (23) would imply that $u_i = 0$ for every i. Likewise we can not have simultaneously $v_k = 0$, $v_l = 0$. Let u_k be one of the two quantities u_k, u_l which is different from zero. Then if we suppose that $u_kv_l - u_lv_k = 0$ we have $v_k = ru_k$ and $v_l = ru_l$ for an appropriate constant r; whence it would follow through use of (23) that $v_i = ru_i$ for every i, so that the solutions u_i and v_i would be dependent, contrary to hypothesis. Therefore, if $\Delta_{kl} \neq 0$ we have $u_kv_l - u_lv_k \neq 0$.

Suppose next that $\Delta_{kl} = 0$. Substituting in (23) the solutions u_i and v_i, we have

$$\pm \Delta_{ml}u_k \pm \Delta_{km}u_l = 0, \qquad \pm \Delta_{ml}v_k \pm \Delta_{km}v_l = 0.$$

Then, if $u_kv_l - u_lv_k \neq 0$, we have $\Delta_{ml} = 0 = \Delta_{km}$ for every m of the set $1, 2, \cdots, n + 2$, except k and l, contrary to hypothesis. Hence, when $\Delta_{kl} = 0$ we have $u_kv_l - u_lv_k = 0$.

This completes the proof of the statement in consideration.

The foregoing result may readily be extended to the case of a system of rank n of n equations in $n + h$ unknown quantities, where h is any integer not less than 2. If a linearly independent set of solutions is $u_{1i}, u_{2i}, \cdots, u_{ki}$ we compare the determinant $|u_jk_i|$ whose element in jth row and ith column is u_jk_i with the determinant $\Delta_{k_1k_2\cdots k_h}$ of the matrix of order n remaining when one strikes from the matrix of coefficients of the original system the columns with numbers $k_1, k_2, \cdots, k_h$. For a given set $k_1, k_2, \cdots, k_h$ these determinants are both zero or neither zero, under conditions similar to those named in the preceding case.

If we replace x in (23) by u and by v we obtain readily from the resulting equations the relations

$$\Delta_{kl}\begin{vmatrix} u_m & u_l \\ v_m & v_l \end{vmatrix} = \pm\Delta_{ml}\begin{vmatrix} u_k & u_l \\ v_k & v_l \end{vmatrix}; \tag{24}$$

and, in particular, the relations

$$\Delta_{k,\,k+1}\begin{vmatrix} u_{k+1} & u_{k+2} \\ v_{k+1} & v_{k+2} \end{vmatrix} = \Delta_{k+1,\,k+2}\begin{vmatrix} u_k & u_{k+1} \\ v_k & v_{k+1} \end{vmatrix}, \qquad k = 1, 2, \cdots, n.$$

Hence, if $\Delta_{i,\,i+1}$ for a given range of consecutive values of the integer is i of one sign (excluding the value zero), then the determinant w_i,

$$w_i = \begin{vmatrix} u_i & u_{i+1} \\ v_i & v_{i+1} \end{vmatrix},$$

is of one sign for every i of the same range. It is obvious that this result also may be appropriately extended to the more general case mentioned in the preceding paragraph.

Let us now prove the following theorem:

Suppose that $\Delta_{i,\,i+1}$ for a given range R of consecutive values of the integer i is of one sign (and hence not zero). Let k and l be two integers of this range, and suppose that u_k and u_{k+1} are not of the same sign and that if either of them is zero it is u_k (in which case u_{k+1} is certainly different from zero). Suppose again that u_l and u_{l+1} are not of the same sign, and that l is the smallest value of the subscript greater than k for which this is so. Then an integer h exists, $k \leqq h \leqq l$, such that v_h and v_{h+1} are not of the same sign. Moreover, there are not two such values of h except possibly for the pair $h = k$, $h = l$.

The last statement follows readily from the preceding part of the paragraph; for, if it were not true, the earlier part of the theorem could be applied with u and v interchanged so as to show that the consecutive character of u_k and u_l is not maintained.

For the given range R of values of i the determinant w_i is of one sign, as we have already seen. Without loss of generality we may suppose that $w_i > 0$, since if it were less than zero we should merely have to change the sign of every u_i or of every v_i. Then we have

$$u_i v_{i+1} > u_{i+1} v_i.$$

Let us first consider the case in which $u_k = 0$. Then v_k is of the opposite sign to u_{k+1}. Without loss of generality in argument we may (and we will) take u_{k+1} positive and v_k negative. Then $u_{k+1}, \cdots, u_l$ are all positive while u_{l+1} is zero or negative. If v_l is zero or negative we see from the inequality $u_l v_{l+1} > u_{l+1} v_l$ that v_{l+1} is positive. Hence either v_l or v_{l+1} is positive. Hence there is a change of sign in the sequence $v_k, v_{k+1}, \cdots, v_{l+1}$ so that the statement in consideration is true for the case when $u_k = 0$.

If $u_k \neq 0$ we may take it to be negative. Then $u_{k+1}, \cdots, u_l$ are positive and u_{l+1} is zero or negative. From the inequality $u_k v_{k+1} > u_{k+1} v_k$ we see that if v_k is positive then v_{k+1} is negative and the truth of the statement in consideration is granted. It remains to consider the case when v_k is negative. Either the statement is verified or we have $v_{k+1}, \cdots, v_l$ all negative. In the latter case we see from the relation $u_l v_{l+1} > u_{l+1} v_l$ that the second member is zero or positive and hence that v_{l+1} is positive. Hence v_l and v_{l+1} are of different signs and the statement is verified in this case.

This completes the proof of the statement in consideration.

Let us now consider the relative distribution of the zeros of the functions $u(s)$ and $v(s)$ gotten from the set of constants u_i and v_i by linear interpolation after the method of § 5. As in the immediately preceding discussion we confine attention to a sequence of consecutive intervals corresponding to a range R of consecutive values of the subscript i for which $\Delta_{i,\,i+1}$ is of one sign. From the results just proved in the foregoing paragraphs it follows that there is a zero of $v(s)$ between two consecutive zeros of $u(s)$ except possibly for the case when the h of these paragraphs has the value k or l.

Let us consider the case when $h = k$. We have

$$u_k v_{k+1} > u_{k+1} v_k,$$

while v_{k+1} and v_k are not of the same sign and are not both zero. If $u_k = 0$ we have a case when $v(s)$ has a zero between the two given consecutive zeros of $u(s)$. Then consider the case when $u_k \neq 0$. Then u_k and u_{k+1} are of different signs; and we may take the former to be negative and the latter positive without loss of generality in argument, and this we do. If in the interval joining the points corresponding to k and $k + 1$ the zero of $v(s)$ is to the right of that of $u(s)$ we have a zero of $v(s)$ between the two consecutive zeros of $u(s)$. We then consider further the other case, namely, that in which the zero of $v(s)$ in the interval in question is to the left of that of $u(s)$. In this case $v_{k+1} \neq 0$. If v_{k+1} is positive we have

$$\frac{u_k}{u_{k+1}} > \frac{v_k}{v_{k+1}},$$

which is contrary to the present hypothesis that the zero of $v(s)$ in the interval in question is to the left of that of $u(s)$. Hence we must now have v_{k+1} negative. Then either $v(s)$ vanishes between the two consecutive zeros of $u(s)$ or v_l is negative. We consider the latter possibility. Now u_l is positive. Hence from the relation $u_l v_{l+1} > u_{l+1} v_l$ we have

$$\frac{v_{l+1}}{v_l} < \frac{u_{l+1}}{u_l}.$$

From this and the fact that u_l and u_{l+1} are not of the same sign it follows readily that the zero of $v(s)$ in the interval from l to $l+1$ is to the left of that of $u(s)$ in the interval, and hence in this case that a zero of $v(s)$ occurs between two consecutive zeros of $u(s)$. We have now exhausted all the possibilities for the case when $h = k$ and have found that in this case a zero of $v(s)$ certainly occurs between two consecutive zeros of $u(s)$.

The case when $h = l$ may be treated in a similar way, with the same conclusion.

From this it follows that there is always at least one zero of $v(s)$ between two consecutive zeros of $u(s)$ on the interval in consideration. There can not be more than one; for, if so, the result could be applied with u and v interchanged so as to lead to the conclusion that $u(s)$ should have a zero between two consecutive zeros of the same function.

The result thus obtained may be formulated into the following theorem:

THEOREM. *Let* $\Delta_{i,\, i+1}$ *for a given range* R *of consecutive values of the integer* i *be of one sign and let* I *denote the interval of the* s*-axis corresponding to this range of* i *in the sense of the treatment in* § 5. *Let* u_i *and* v_i *be two linearly independent solutions of system* (22); *and let these solutions be extended by the method of linear interpolation employed in* § 5 *to the functions* $u(s)$ *and* $v(s)$. *Then on the interval* I *the zeros of* $u(s)$ *and* $v(s)$ *separate each other.*

Let us now consider more generally a system of equations

$$\sum_{j=1}^{n+h} a_{ij}x_j = 0, \qquad i = 1, 2, \cdots, n, \tag{25}$$

in the $n+h$ unknown quantities $x_1, x_2, \cdots, x_{n+h}$, where $h \geqq 2$; and let us suppose that the matrix of this system is of rank n. The system then has h linearly independent solutions; we designate such a set by

$$x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(h)}.$$

Let D_i be the determinant of order n of the square matrix gotten from the matrix of coefficients in (25) by striking out h consecutive columns beginning with the ith; let R be a range of consecutive values of the subscript i for which D_i is of one sign (zero values being excluded); and let I denote the interval of the s-axis corresponding to this range of i in the sense of the treatment in § 5.

Now let w_i denote the determinant

$$w_i = \begin{vmatrix} x_i^{(1)} & x_{i+1}^{(1)} & \cdots & x_{i+h-1}^{(1)} \\ x_i^{(2)} & x_{i+1}^{(2)} & \cdots & x_{i+h-1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ x_i^{(h)} & x_{i+1}^{(h)} & \cdots & x_{i+h-1}^{(h)} \end{vmatrix} = \begin{vmatrix} x_i^{(1)} & \Delta x_i^{(1)} & \cdots & \Delta^{h-1}x_i^{(1)} \\ x_i^{(2)} & \Delta x_i^{(2)} & \cdots & \Delta^{h-1}x_i^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ x_i^{(h)} & \Delta x_i^{(h)} & \cdots & \Delta^{h-1}x_i^{(h)} \end{vmatrix}, \tag{26}$$

where $\Delta, \Delta^2, \cdots, \Delta^{h-1}$, in the last form of the determinant, denote the first, second, $\cdots$, $(h-1)$th differences of the x_i with respect to the variable i. It is obvious that the two forms of the determinant have the same value.

Since D_i is of one sign for i on R it follows from the earlier work in this section that w_i is of one sign for i on R. Let us consider the matrix of any m rows of the first [second] determinant form for w_i in (26) and let $A_{imj}[C_{imj}]$, $j = 1, 2, \cdots, j_m$, be all the m-rowed determinants formed from this matrix. Let $B_{imj}[D_{imj}]$, $j = 1, 2, \cdots, j_m$, be the corresponding algebraic complements. By means of the Laplace development for determinants we then have

$$w_i = \sum_{j=1}^{j_m} A_{imj}B_{imj} = \sum_{j=1}^{j_m} C_{imj}D_{imj}. \tag{27}$$

Suppose that k is a value of i on the range R such that B_{imj} $[D_{imj}]$ is of one sign or zero for each j of the set $j = 1, 2, \cdots, j_m$; and suppose that l is a value of i also on R for which these quantites are of the opposite sign or zero. Then, since w_i is of one sign for i on R, it can not be true that all of the quantities A_{kmj}, A_{lmj} $[C_{kmj}, C_{lmj}]$ are of one sign for the fixed value of m and for j varying over its whole range or indeed over just that part of its range for which the corresponding B_{kmj} or B_{lmj} $[D_{kmj}$ or $D_{lmj}]$ has a value different from zero.

In particular, if B_{kmj} and B_{lmj} $[D_{kmj}$ and $D_{lmj}]$ both have the value zero except for a single value ρ of j and for this value have different signs, then $A_{km\rho}$ and $A_{lm\rho}$ $[C_{km\rho}$ and $C_{lm\rho}]$ have opposite signs. Hence the function $A_{sm\rho}$ $[C_{sm\rho}]$ obtained from $A_{im\rho}$ $[C_{im\rho}]$ by linear interpolation as in § 5 has a zero on that part of the s-axis which corresponds to the range of i from k to l inclusive.

Let us now consider the first determinant in (26) for the case when m is unity and the one row singled out is the last one. Then the A_{i1j}, for varying j, are the elements of the last row of the determinant and the B_{i1j} are their cofactors. If for $i = k$ these cofactors are of one sign or zero and for $i = l$ are of the opposite sign or zero, where k and l are both on the range R of i, then not all of the quantities $x_k^{(h)}$, $x_{k+1}^{(h)}$, $\cdots$, $x_{l+h-1}^{(h)}$ are of one sign. Hence if we form the function $x^{(h)}(s)$ by linear interpolation from $x_i^{(h)}$ this function $x^{(h)}(s)$ has at least one zero in the interval corresponding to the range of i from k to $l+h-1$. Thus for the cases when our results apply we are able by means of $h-1$ linearly independent solutions of (25) to define an interval for i and a corresponding interval on the s-axis such that every solution $x_i^{(h)}$ of (25) which is linearly independent of the first $h-1$ solutions gives rise by linear interpolation to a function $x^{(h)}(s)$ which vanishes on the interval of the s-axis in question. It will be observed that this result is a direct generalization of a portion of that obtained above for the case when $h = 2$.

Let us consider the second determinant in (26) for the case when m is unity and the one row singled out is the last one. Then the C_{i1j}, for varying j, are the elements of the last row and the D_{i1j} are their cofactors. If the latter for $i = k$ are of one sign or zero and for $i = l$ are of the opposite sign or zero, where k and l are values of i on the range R, then the quantities $x_i^{(h)}, \Delta x_i^{(h)}, \cdots, \Delta^{h-1} x_i^{(h)}$, for the two values k and l of i, can not all be of the same sign. In particular, if they are all positive for $i = k$, then one at least of them must fail to be positive at $i = l$; so that at least one of them changes sign or becomes zero in the interval in question.

We may look upon the modification of the first determinant in (26) to produce the second as a single instance of a great variety of transformations of that first determinant, each being made by replacing columns in that determinant by various linear homogeneous combinations of columns (the coefficients being constants or functions of i) in such wise that the non-vanishing of w_i on a range R implies the non-vanishing of the modified determinant $\bar{w}_i$ on the same range R. For every such $\bar{w}_i$ we may obtain a sequence of theorems of the sort just brought to light in the treatment of the two forms given in (26). Such properties as are thus indicated will be useful to us later in the study of certain transcendental problems which are limiting cases of the algebraic problems now in consideration.

Solutions of (25), for a given value of $h > 2$, may also be studied by means of similar systems for a smaller value of h, this smaller value of h being not less than 2. For this purpose we adjoin to (25) one or more additional equations in the same unknown quantities, these equations being linearly independent of each other and of those in (25). In particular, the system may thus be replaced by a new system of the same form but with only two linearly independent solutions. The foregoing results for such a system, when applied to this, yield properties of those solutions of (25) which satisfy the auxiliary conditions, the latter being somewhat analogous to boundary conditions for differential equations. A considerable hold can thus be gotten on a class of solutions of (25). This obvious remark will serve a useful heuristic purpose in studying certain properties of differential equations with boundary conditions.

In this section we have confined ourselves principally to that part of the range of i for which D_i is of one sign. We propose the problem of investigating the matter for the whole range of i from 1 to n and thus of associating the zeros of D_i and of certain combinations of the solutions of (25) for the whole range of i from 1 to n.

7. *General Character of a Fundamental Limiting Operation.*—It is well known, as we have already pointed out more fully, that several important types of functional equations, such as linear differential equations, linear

integral equations, etc., can be exhibited as limiting cases of algebraic systems. In a large range of limiting cases the conjugate character of the solutions of (1) and (2) which we have discussed in § 3 and the expansion by use of conjugate sets treated in § 4 are carried over to the limiting problems in such a way as to afford a definite suggestion of properties to be expected and of methods of analysis for examining them. We shall now consider the general character of the limiting operation in question with reference to the properties just indicated.

The graphic representations employed in § 5 afford a starting-point. Let us take a fixed interval $a \leqq s \leqq b$ of the real s-axis and let us associate with certain points of this interval the discrete values of i and j in equations (1) and (2). For the hth equation in either system we take on (ab) a set of points n_h in number including the points a and b; and we interpolate x_{hj} into a function $x_h(s)$ by linear interpolation as in § 5. Similarly, a function a_{khij} is interpolated into a function $a_{kh}(s, t)$ by a method also described in § 5. Equations (1) and (2) may now be looked upon as establishing relations among the functional values of these functions of s and t at certain points only of the axes of s and t; and so of defining the solution functions at these points and these alone, their definitions being completed by the method of interpolation agreed upon.

The limiting processes described consist essentially in this: let the various numbers n_h (or at least a part of them) increase indefinitely; and as they increase let the functions $x_h(s)$ pass through a corresponding sequence of changes. Various limiting processes are thus set up in the original equations and in their solutions. If these lead to replacing the original equations by well-defined equations and their solutions by well-defined functions, we have in the process a suggestion of a heuristic guide to probable solutions of the limiting problems and to certain probable fundamental properties of these. As we proceed it will be seen that this guide has great usefulness in a considerable range of problems.

We shall usually require that the distribution of basic points on the interval (ab) of the s-axis shall undergo change in such wise that the maximum distance between two consecutive points shall approach zero as n_h becomes infinite. This suggests that the sum as to j in (1) and (2) may sometimes be replaced by an integral as to t from a to b; and, when this is to be done, it is desirable that the coefficients a_{khij} shall be replaced each by a product one factor of which is to correspond to the differential in the limiting case. [If the basis points on the s-axis (or t-axis) are equally spaced one may introduce this needed factor simply by multiplying the equation through by Δs (or Δt), the distance between two consecutive basic points. This is perhaps the easiest way to make the modification in question.] In fact such a form has frequently been employed in setting up a limiting process to guide one to the theory of integral equations, first by Volterra and later by others. In numerous ways (of which that just mentioned is one of the most important) one may subject the coefficients in (1) and (2) to subsidiary conditions such that the limiting process in consideration shall lead to desired forms of functional equations as limiting cases, whether differential equations or integral equations or equations of other type.

The several systems in a single set of systems (1) or (2) are independent of each other in such wise that the limiting processes may be applied to each with a certain measure of independence of the others, so that we might have for instance an ordinary differential equation and an integral equation to make up the limiting system of two equations for the case of two parameters λ. Numerous other combinations are logically possible; and several of them are interesting for the results to which they give rise. In a second paper we shall give a precise characterization of several problems to which we are led in carrying out the limiting operation in a variety of ways or combinations of ways.

It remains here to note certain characteristic results for the finite case which persist (in modified form of course) in a great variety of limiting cases.

The property of conjugacy expressed by the equations of system (8) is an important one which persists after certain of the limiting processes just indicated have been carried out. It takes a large number of forms owing to the great variety of ways in which the limiting process may be set up. One of the simplest and at the same time most useful of these is that in which the determinant $D_{i_1 j_1 \dots i_r j_r}$ in (8) becomes a function $D(s_1, t_1, \cdots, s_r, t_r)$ and the summation in (8) is replaced by an integration, the algebraic system having first been taken in a form which is convenient for passing to the limit in a manner to realize such a result. The property of conjugacy may then be expressed by the relations

$$(28) \quad \int_a^b \int_a^b \cdots \int_a^b D(s_1, t_1, \cdots, s_r, t_r) \cdot \prod_{h=1}^{r} x_h^{(\rho)}(s_h) y_h^{(\sigma)}(t_h) \cdot ds_1 dt_1 \cdots ds_r dt_r = 0, \qquad \rho \neq \sigma.$$

For a wide range of cases we have in the limiting problem an infinite number of sets of characteristic values for the parameters λ and hence an infinite number of sets of functions $x_h^{(\rho)}(s)$, $y_h^{(\rho)}(t)$, $h = 1, \cdots, r$, for varying ρ, for which relation (28) is valid.

One of the most frequently occurring and important special cases of (28) is that which corresponds to equation (12), namely,

$$\int_a^b b(s) x^{(\rho)}(s) y^{(\sigma)}(s) ds = 0, \qquad \rho \neq \sigma.$$

When we proceed in the second paper to certain important instances of the theory here sketched roughly in a general sort of way we shall find that equations essentially like (28) or special cases of it appear in a great variety of forms; but a little examination of each system will show that its equations arise in a natural way as limiting cases of the equations in (8) and that one is led to them by the guide afforded by the general considerations barely outlined here.

Now if the repeated integral in the first member of (28) has a value different from zero when $\rho = \sigma$ (a condition which is realized in a wide range of important cases, as we shall see) and if a function $f(s_1, s_2, \cdots, s_r)$ has an expansion of the form

$$f(s_1, s_2, \cdots, s_r) = \sum_{\rho=1}^{\infty} c_\rho x_1^{(\rho)}(s_1) x_2^{(\rho)}(s_2) \cdots x_r^{(\rho)}(s_r),$$

where the c_ρ are constants and the values of ρ correspond to the various sets of characteristic values of the parameters λ, then the coefficients c_ρ are readily found by a method analogous to that by which the Fourier coefficients for a function are found; and we have in fact

$$c_\rho = \frac{\begin{array}{r}\displaystyle\int_a^b \int_a^b \cdots \int_a^b f(s_1, s_2, \cdots, s_r) D(s_1, t_1, \cdots, s_r, t_r) y_1^{(\rho)}(t_1) y_2^{(\rho)}(t_2) \\ \cdots y_r^{(\rho)}(t_r) ds_1 dt_1 \cdots ds_r dt_r\end{array}}{\displaystyle\int_a^b \int_a^b \cdots \int_a^b D(s_1, t_1, \cdots, s_r, t_r) \cdot \prod_{h=1}^{r} x_h^{(\rho)}(s_h) y_h^{(\rho)}(t_h) \cdot ds_1 dt_1 \cdots ds_r dt_r}.$$

In each instance of our theory this formal expansion or the requisite modification of it is used to set the basic expansion problem associated with the instance in consideration.

In the foregoing discussion the number of variables in the function f to be expanded is equal to the number of parameters in the algebraic problem from which we passed to the transcendental problem. This correspondence is not essential to the nature of the process involved. The desired extension can best be brought out by starting from the particular expansion (19). We may carry out our method of linear interpolation as at the close of § 5 and obtain the expansion (21). If now we proceed to the limiting case in such wise that μ and ν simultaneously approach infinity and if we require that the distribution of axis points on the s-axis and the t-axis shall have the usual property of condensation and if the number of distinct characteristic values approaches infinity during the process, we shall be led heuristically to an expansion of the form

$$f(s, t) = \sum_{k=1}^{\infty} c_k x^{(k)}(s, t),$$

where the x's are now solutions of the limiting problem. It is easy to see that the properties of conjugacy are maintained (under appropriate circumstances) and that formulæ are therefore easily found for the values of the c_k. We do not attempt to investigate in general the range of validity of these results. We shall later find in them a useful heuristic guide to problems in partial differential equations; and for the cases involved in such applications we shall analyze the matter of range of validity.

In a similar way we may obtain numerous modifications of the foregoing expansion of $f(s_1, s_2, \cdots, s_h)$ by starting from the modifications of equations (14) suggested by the method of the preceding paragraph for the simplest special case and proceeding in a similar manner.

Another method of proceeding from (19) to a limiting case will be very useful to us in dealing with a system of ordinary linear differential equations. In this method we suppose that the passage to the limit is effected in such way that μ remains fixed so that i is always a discrete variable ranging over the finite set $1, 2, \cdots, \mu$, while ν becomes infinite and the variable j passes in our usual way to the continuous variable s in the limiting equation. Then expansion (19) passes over into the form

$$f_i(s) = \sum_{k=1}^{\infty} c_k x_i^{(k)}(s), \qquad i = 1, 2, \cdots, \mu,$$

a form which may be looked upon as affording the simultaneous expansion of μ given functions $f_i(s)$ of a continuous variable s in terms of μ sets of functions $x_i(s)$, the coefficients c_k being the same in each of the two expansions.* Obviously, numerous more general forms of like expansions of systems of functions of one or more variables may be obtained by passing from the general relation (14) to limiting cases in the various ways sufficiently suggested in the foregoing instances.

The details of the matters suggested in the three preceding paragraphs will be developed only in connection with the instances of certain transcendental problems to which they afford the requisite guide.

It is doubtless true that many other properties of the algebraic systems can be carried over to certain of the important limiting cases. In fact certain of the zero separation theorems of § 6 can be utilized to suggest like theorems for the solutions of certain transcendental problems. Moreover, the interaction of the two types of problems may well be made to work in both directions. A part of the results in § 6 were indeed suggested by classic properties of the solutions of linear differential equations of the second order. Once in hand for the simpler cases these results were easily extended to other cases of algebraic systems and then afforded a new guide to certain related transcendental problems. There is thus a repeated mutual reaction between the two types of problems which is useful in the development of the theory of each.

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* A special case of expansions of this sort first came to my attention in the dissertation of Dr. C. C. Camp (a manuscript copy of which he was kind enough to lend me); and the suggestion afforded by it has been very useful to me in the present investigation.

8. *A Second Type of Limiting Case.*—Let us now consider the r homogeneous linear systems in the infinite number of unknown quantities x,

$$(29) \quad \sum_{j=1}^{\infty} (a_{0hij} + \lambda_1 a_{1hij} + \cdots + \lambda_r a_{rhij}) x_{hj} = 0, \qquad i = 1, 2, 3, \cdots,$$

a separate system being formed for each value h of the set $1, 2, \cdots, r$. With these systems let us associate the r adjoint systems

$$(30) \quad \sum_{j=1}^{8} (a_{0hji} + \lambda_1 a_{1hji} + \cdots + \lambda_r a_{rhji}) y_{hj} = 0, \qquad i = 1, 2, \cdots.$$

Here $\lambda_1, \lambda_2, \cdots, \lambda_r$ are r parameters by the choice of which the simultaneous consistency of the r systems in either set is to be secured. For these infinite systems a great variety of possibilities may be realized by special methods of defining the coefficients a; and a comprehensive analysis of all mutual possibilities would require the examination of a great many cases. Consequently we shall confine our attention to a range of cases in which the theory may be set forth in the simplest manner.

A set of values of $\lambda_1, \cdots, \lambda_r$ for which (29) [(30)] has a solution not identically zero will be called *a set of characteristic values for* (29) [(30)]. In the case of the finite problem of § 2 we saw that the sets of characteristic values for the two adjoint problems were identical. For the infinite case of this section we shall consider only those systems for which this property is maintained. It is clearly maintained in cases in which those characteristic values can be determined by determinantal equations analogous to equations (3) of § 2. If this method of determining the sets of characteristic values is not suitable we may proceed as follows for a certain range of cases of equations (29) and (30); omit from each system r equations, one for each value of h; solve the remaining equations as non-homogeneous equations, after having assigned a value to one of the unknown quantities (an operation which can be carried out by known processes in a wide range of cases), and suppose that this solution is unique (as indeed it is in a wide range of known cases); now substitute into the equation omitted from each system the values of the unknown quantities so obtained; thus we have for system (29), and also for system (30), r equations involving the r parameters; their solutions give the sets of characteristic values.

Now, we shall suppose that the sets of characteristic values, however determined (whether by one of the methods indicated or by another), is denumerably infinite and we shall denote them and the corresponding solutions by

$$\lambda_k^{(\rho)}, \quad x_{hi}^{(\rho)}, \quad y_{hi}^{(\rho)}, \qquad \rho = 1, 2, 3, \cdots.$$

Several of the properties developed for the corresponding algebraic problem are now readily carried over to these solutions of a limiting problem, at least in a formal way. Since our present interest in the general procedure arises from intended applications of it to relatively simple instances we shall not undertake an analysis of the range of validity of our results, but shall content ourselves with little more than an exhibition of the formal processes.

If we proceed by a method in all respects similar to that employed in § 3 and if we have before us a case in which the order of infinite summations may be interchanged in ways now needful in carrying out the same method, we come through to certain equations analogous to (8) and expressing the fundamental conjugate character of the solutions of (29) and (30), namely, the equations

$$(31) \qquad \sum_{i_1=1}^{\infty} \sum_{j_1=1}^{\infty} \cdots \sum_{i_r=1}^{\infty} \sum_{j_r=1}^{\infty} D_{i_1j_1\cdots i_rj_r} \prod_{h=1}^{r} x_{hi_h}^{(\rho)} y_{hj_h}^{(\sigma)} = 0, \qquad \rho \neq \sigma,$$

where $D_{i_1j_1\cdots i_rj_r}$ is notationally the same determinant as that which is represented by this symbol in equation (8).

Following still the method of § 3 it would be possible to name conditions on the coefficients a which are sufficient to insure that the first member of (31) shall have a value different from zero when $\rho = \sigma$. Since we have no need for the explicit form of these conditions we shall not take the space to derive them; we shall merely assume that we have before us a case in which the first member of (31) for $\rho = \sigma$ is always different from zero.

It is obvious that we have particularly simple forms of the conjugacy conditions (31) for cases corresponding to those involved in equations (11), (12), (13); and that we may have modifications of all of the conjugacy conditions analogous to those indicated briefly for the finite case in the paragraph following equation (13).

We are now led to consider the problem of the expansion of a function $z_{i_1i_2\cdots i_r}$ of r discrete arguments, each on the range 1, 2, 3, $\cdots$, in terms of the solutions of one set of systems, the expansion being of the form

$$(32) \qquad z_{i_1i_2\cdots i_r} = \sum_{k=1}^{\infty} c_k x_{1i_1}^{(k)} x_{2i_2}^{(k)} \cdots x_{ri_r}^{(k)},$$

where the coefficients c_k are independent of the variables $i_1, \cdots, i_r$. If the function z has an expansion of this form and if certain changes of order of summation involved in the obvious process of reckoning are legitimate, we obtain readily for the coefficients c_k the values

$$(33)\quad c_k = \frac{\sum_{i_1=1}^{\infty}\sum_{j_1=1}^{\infty}\cdots\sum_{j_r=1}^{\infty}\sum_{j_r=1}^{\infty} D_{i_1j_1\cdots i_rj_r} z_{i_1i_2\cdots i_r}\prod_{h=1}^{r} y_{hj_h}^{(k)}}{\sum_{i_1=1}^{\infty}\sum_{j_1=1}^{\infty}\cdots\sum_{j_r=1}^{\infty}\sum_{j_r=1}^{\infty} D_{i_1j_1\cdots i_rj_r}\prod_{h=1}^{r} x_{hi_h}^{(k)} y_{hj_h}^{(k)}},\qquad k = 1, 2, 3, \cdots.$$

The discussion associated with equations (17) and (18) affords suggestions for the formation of special cases or for modifying the form of expansion in (32) and (33).

Even in this transcendental case one may usefully employ a graphic representation of the sort described in § 5. One probably takes most naturally for an s-axis the real axis from 1 to infinity and erects perpendiculars at the points 1, 2, 3, $\cdots$ on this axis, these perpendiculars to serve as the axes of coördinates. One may then fill in the function by linear interpolation as in § 5 and so make a function of a continuous variable. One might conceivably carry out now a limiting process by which the number of perpendiculars in any finite stretch increases indefinitely and the greatest distance between two which are consecutive decreases to zero. One has a limiting case which may be employed to afford suggestions for dealing with differential equations of infinite order. Various modifications and extensions of this graphic representation are suggested by the foregoing treatment for the finite case, particularly that in § 7; but we shall not now pursue the matter further.

Besides the generalization of the problem associated with (1) and (2) to that associated with (29) and (30) one might treat also that in which the number of parameters λ becomes infinite while the number of x's and hence of y's remains finite or that in which the number of λ's and the number of x's and y's simultaneously become infinite; but it appears unlikely that such generalizations will be as useful as that which is indicated in the foregoing paragraphs.

A rather important problem, upon the resolution of which we shall not enter, may be suggested here, namely, the problem of the relative distribution of the zeros of linearly interpolated solutions of infinite systems.

9. *Variation of Parameters and Green's Functions for Algebraic Systems.* —Let us consider a system of n non-homogeneous linear algebraic equations

$$(34)\qquad \sum_{j=1}^{n+h} a_{ij}u_j = b_i,\qquad i = 1, 2, \cdots, n,$$

and the corresponding homogeneous system

$$(35)\qquad \sum_{j=1}^{n+h} a_{ij}x_j = 0,\qquad i = 1, 2, \cdots, n,$$

each in $n + h$ unknown quantities, h being a positive integer, and let us suppose that the rank of the matrix $||a_{ij}||$ is n.

In what follows next we shall assume that the notation is chosen so that the determinant of the square matrix made up of the first n columns of $||a_{ij}||$ has a value different from zero. If the equations in each system are taken in a suitable order (and we shall suppose them to be so written already) it is obviously possible to combine them into a new equivalent system in the same unknown quantities in such way that the new coefficients a_{ij} have the value zero when $j < i$. We shall now suppose further that the original, and hence the new, matrix $||a_{ij}||$ has the property that the determinants of order 1, 2, 3, $\cdots$, n in its lower right-hand corner are all different from zero in value. Then it is possible to make further combinations of equations in either system so as to replace the system by an equivalent system in the same unknown quantities and in such wise that when all this is done the systems (34) and (35) take the forms

$$(36) \qquad \sum_{r=0}^{h} \alpha_{i,\ i+r} u_{i+r} = \beta_i, \qquad \sum_{r=0}^{r} \alpha_{i,\ i+r} x_{i+r} = 0, \qquad i = 1, 2, \cdots, n,$$

where $\alpha_{i,\ i}$ and $\alpha_{i,\ i+h}$ are different from zero for every i. For the equations in this normal form we have a ready method of obtaining the solutions of the non-homogeneous system in terms of a fundamental system of solutions of the homogeneous system analogous to, and indeed abstractly identical with, the classic method of variation of parameters in the theory of differential equations, the latter being in fact a limiting case of the former. This method for the algebraic systems we shall now develop in a brief treatment.

Let us denote by

$$x_j^{(1)}, \quad x_j^{(2)}, \quad \cdots, \quad x_j^{(h)}, \qquad j = 1, 2, \cdots, n + h,$$

a fundamental system of h linearly independent solutions of the homogeneous system in (36), and hence of that in (35). Then we seek a solution of the non-homogeneous system in (36), and hence of that in (34), in the form

$$(37) \qquad \begin{gathered} u_{i+t} = c_i^{(1)} x_{i+t}^{(1)} + c_i^{(2)} x_{i+t}^{(2)} + \cdots + c_i^{(h)} x_{i+t}^{(h)}, \\ t = 0, 1, \cdots, h - 1, \qquad i = 1, 2, \cdots, n + 1, \end{gathered}$$

where the functions $c_i^{(k)}$ are to be determined. The equations in the system

$$(38) \qquad \begin{gathered} \sum_{k=1}^{h} [c_{i+1}^{(k)} - c_i^{(k)}] x_{i+1+t}^{(k)} = 0, \qquad t = 0, 1, \cdots, h - 2, \quad i = 1, 2, \cdots, n, \\ \sum_{k=1}^{h} [c_{i+1}^{(k)} - c_i^{(k)}] x_{i+1}^{(k)} = \frac{\beta_i}{\alpha_{i,\ i+h}}, \qquad i = 1, 2, \cdots, n, \end{gathered}$$

are now obtained as follows: those in the first line come from (37) by comparison of that equation for a given value $i + 1$ of i with the equation gotten from (37) when t is replaced by $t + 1$; those in the last line are obtained by substituting the value of u from (37) into the non-homogeneous system of (36), the value of u_{i+t} for $t = 0, 1, \cdots, h - 1$ being written as in (37) while that of u_{i+h} is taken in the form

$$u_{i+h} = c_i^{(1)} x_{i+h}^{(1)} + \cdots + c_i^{(h)} x_{i+h}^{(h)} + \sum_{k=1}^{h} [c_{i+1}^{(k)} - c_i^{(k)}] x_{i+h}^{(k)}.$$

We look upon equations (38) as furnishing a system for the determination of the functions $c_i^{(k)}$.

The condition that (38), for every value i of the set $1, 2, \cdots, n$ and for every β_i, shall serve for the determination of the quantities in the square brackets is that the determinant Δ_i,

$$\Delta_i = \begin{vmatrix} x_{i+1}^{(1)} & x_{i+1}^{(2)} & \cdots & x_{i+1}^{(h)} \\ x_{i+2}^{(1)} & x_{i+2}^{(2)} & \cdots & x_{i+2}^{(h)} \\ \cdot & \cdot & \cdot & \cdot \\ x_{i+h}^{(1)} & x_{i+h}^{(2)} & \cdots & x_{i+h}^{(h)} \end{vmatrix},$$

shall be different from zero for every i of the set $1, 2, \cdots, n$. But in § 6 we saw that these inequalities are equivalent to the conditions that the determinant of every matrix of order n obtained from $||a_{ij}||$ by striking out h consecutive columns shall be different from zero. These are then necessary conditions for it to be true that every system (34) may be solved by the given method of variation of parameters in terms of a fundamental system of solutions of (35). These conditions, together with those already named in connection with the reduction to the normal systems (36), are also sufficient conditions for it to be true that every system (34) may be solved by the given method, as we shall now show by actually effecting the solution under the named hypotheses.

If we denote by $\Delta_i^{(k)}$ the cofactor of the element in the kth column and last row of Δ_i we have from (38) the relations

$$(39) \qquad c_{i+1}^{(k)} - e_i^{(k)} = \frac{\beta_i}{\alpha_{i,\ i+h}} \frac{\Delta_i^{(h)}}{\Delta_i}, \qquad k = 1, 2, \cdots, h; \qquad i = 1, 2, \cdots, n.$$

If we assign to $c_1^{(k)}$, or to $c_{n+1}^{(k)}$, an arbitrary value $c^{(k)}$, the equations (39) serve by recursion to determine uniquely the $c_i^{(k)}$ in terms of these arbitrary values; and in fact $c_i^{(k)}$ for fixed k and varying i is the arbitrary element $c^{(k)}$ plus a determinate function of i depending only on the b_i, the a_{ij} and the solutions of the homogeneous system. On substituting these values of the c's into (37) we have the general solution u_j of the non-homogeneous system in (36), and hence of system (34).

This solution of (34) may obviously be written in the customary form

$$(40)\quad u_i = c^{(1)}x_i^{(1)} + c^{(2)}x_i^{(2)} + \cdots + c^{(h)}x_i^{(h)} + \bar{u}_i, \qquad i = 1, 2, \cdots, n+h,$$

where $\bar{u}_i$ is a particular solution of (34) and the $c^{(1)}, \cdots, c^{(h)}$ are arbitrary constants, this form of solution being of course always realized under the hypotheses of the first paragraph of the section.

If we take systems of the form (36) where i runs over the infinite set $1, 2, 3, \cdots$ instead of the finite set $1, 2, \cdots, n$, it is easy to see that we have a process in every respect identical with the foregoing except that the recurrence relations which replace (38) are now to be solved by taking the first $c_i^{(k)}$ only (since there is no last), namely $c_1^{(k)}$, as arbitrarily defined. Such a system is intimately connected with the theory of difference equations.

Let us now consider the question of adjoining linear conditions to the systems (34) and (35), these new conditions to be thought of as being analogous to the boundary conditions associated with differential equations for instance. Let these new conditions be taken in the forms

$$(41)\quad \sum_{j=1}^{n+h} a_{ij}u_j = b_i, \quad \sum_{j=1}^{n+h} a_{ij}x_j = 0, \quad i = n+1, n+2, \cdots, n+s,\ s \leqq h.$$

The question is on the possibility of choice of the $c^{(k)}$ in (40) and of the $\bar{c}^{(k)}$ in the relation

$$x_i = \bar{c}^{(1)}x_i^{(1)} + \cdots + \bar{c}^{(h)}x_i^{(h)}$$

so that systems (41) shall be satisfied; that is, it is on the possibility of the systems

$$(42)\quad \sum_{k=1}^{h} c^{(k)} \sum_{j=1}^{n+h} a_{ij}x_j^{(k)} + \sum_{j=1}^{n+h} a_{ij}\bar{u}_j = b_i, \quad \sum_{k=1}^{h} \bar{c}^{(k)} \sum_{j=1}^{n+h} a_{ij}x_j^{(k)} = 0,$$
$$i = n+1, \cdots, n+s.$$

In case $s < h$ the latter system always has solutions $\bar{c}^{(k)}$. It may also have solutions when $s = h$. When it has solutions the former system may or may not have a solution; its having a solution depends on the character of the $\bar{u}_j$ and hence on the character of the b_i in (34) and also on the character of the new a_{ij} and b_i in (41). If $s = h$ and if the last system in (42) has no solution except that for which the $\bar{c}^{(k)}$ are all zero then the first system in (42) has a unique solution. This last algebraic theorem (essentially obvious in its character) has for an analogue a fundamental theorem in the theory of differential equations (one which is not obvious in character).*

Let us now consider the system

$$(43)\qquad \sum_{j=1}^{n} a_{ij}v_j = c_i, \qquad i = 1, 2, \cdots, n,$$

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* Bôcher, "Les Méthodes de Sturm," pp. 19ff.

where the determinant Δ of the coefficients, $\Delta \equiv |a_{ij}|$, is different from zero. Then the customary solution (which is unique) may be written in the form

$$v_j = \sum_{i=1}^{n} G_{ij} c_i, \qquad j = 1, 2, \cdots, n, \tag{44}$$

where

$$G_{ij} = \frac{\Delta_{ij}}{\Delta},$$

Δ_{ij} being the cofactor of the element in the ith row and jth column of Δ. The function G_{ij} has been called the Green's function* for the homogeneous system $\Sigma a_{ij} v_j = 0$ corresponding to system (43). For fixed i it satisfies every equation of this system except the ith, the first member of the ith equation having for this value $v_j = \Delta_{ij}/\Delta$ the value 1 instead of zero. Thus, by means of the homogeneous system, *which is incompatible,* we can define a Green's function G_{ij} of such sort that the solution of (43) for every set c_i is expressible in the form (44). If the homogeneous system is not incompatible it is easy to see that no function G_{ij} exists having with respect to (43) the named property.

If H_{ij} is the Green's function of the system adjoint to system (43) in the sense of § 2 it is obvious that $H_{ji} = G_{ij}$.

This obvious matter for algebraic systems is mentioned here for its use in suggesting the requisite procedure for analogous properties of certain functional equations to be treated in later papers and especially for its use in transforming the expansions of § 4 into the form of contour integrals in § 10 immediately following.

10. *Condensation of Algebraic Expansions into Contour Integrals.* Let us consider systems (10) with a single parameter λ and suppose that the determinant $\Delta(\lambda)$ of the first system has n distinct simple roots $\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(n)}$ and that to each root corresponds a unque solution of the system (except for a constant factor). Denote by $\Delta_{ij}(\lambda)$ the cofactor of the element in the ith row and jth column of $\Delta(\lambda)$. Then the Green's functions $G_{ij}(\lambda)$ and $H_{ij}(\lambda)$ for the two systems in (10) are respectively

$$G_{ij}(\lambda) = \frac{\Delta_{ij}(\lambda)}{\Delta(\lambda)}, \qquad H_{ij}(\lambda) = G_{ji}(\lambda) = \frac{\Delta_{ji}(\lambda)}{\Delta(\lambda)}.$$

These Green's functions are analytic in the complex variable λ except at the zeros of $\Delta(\lambda)$, that is, except for characteristic values $\lambda^{(k)}$ of systems (10).

Now we may write the Green's function $G_{ij}(\lambda)$ in the forms

$$G_{ij}(\lambda) = \frac{R_{kij}}{\lambda - \lambda^{(k)}} + S_{kij}(\lambda), \qquad k = 1, 2, \cdots, n,$$

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* Bôcher, *Annals of Mathematics,* (2) 13 (1911), pp. 71ff.

where $S_{kij}(\lambda)$ is analytic in λ at $\lambda = \lambda^{(k)}$ and R_{kij} is the residue at $\lambda = \lambda^{(k)}$ of $G_{ij}(\lambda)$ considered as a function of the complex variable λ. We propose to evaluate the residue R_{kij} in terms of the solutions of systems (10).

In the first place, we have

$$R_{kij} = \frac{\Delta_{ij}(\lambda^{(k)})}{\Delta'(\lambda^{(k)})}\,.$$

where $\Delta'(\lambda^{(k)})$ denotes the value at $\lambda = \lambda^{(k)}$ of the derivative of $\Delta(\lambda)$ with respect to λ. The quantities $x_j = R_{k\sigma j}$, for fixed k and σ, afford a solution of the first system in (10) for $\lambda = \lambda^{(k)}$. Likewise, the quantities $y_j = R_{kj\rho}$, for fixed k and ρ, afford a solution of the second system in (10) for $\lambda = \lambda^{(k)}$. But all solutions of these systems for $\lambda = \lambda^{(k)}$ are constant multiples of $x_j^{(k)}$ and $y_j^{(k)}$ respectively. Hence we have

$$R_{kij} = \gamma_k x_j^{(k)} y_i^{(k)},$$

where γ_k is independent of i and j. It remains to evaluate γ_k.

Now we have

$$\lim_{\lambda=\lambda^{(k)}} \{(\lambda - \lambda^{(k)})G_{it}(\lambda) - \gamma_k x_t^{(k)} y_i^{(k)}\} = 0;$$

whence it follows that

$$\text{(45)} \qquad \lim_{\lambda=\lambda^{(k)}} \left\{ (\lambda - \lambda^{(k)}) \sum_{i=1}^{n} \sum_{j=1}^{n} G_{it}(\lambda) b_{ij} x_j^{(k)} - \gamma_k x_t^{(k)} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_j^{(k)} y_i^{(k)} \right\} = 0.$$

But

$$\sum_{j=1}^{n} (c_{ij} + \lambda b_{ij}) x_j^{(k)} = (\lambda - \lambda^{(k)}) \sum_{j=1}^{n} b_{ij} x_j^{(k)},$$

so that from the fundamental property of the Green's function we have

$$x_t^{(k)} = \sum_{i=1}^{n} G_{it}(\lambda)(\lambda - \lambda^{(k)}) \sum_{j=1}^{n} b_{ij} x_j^{(k)}.$$

Putting this value of $x_t^{(k)}$ into (45) we have readily

$$1 - \gamma_k \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_j^{(k)} y_i^{(k)} = 0.$$

With the value of γ_k afforded by this relation we have for the residue R_{kij} of $G_{ij}(\lambda)$ at $\lambda = \lambda^{(k)}$ in the complex plane the value

$$\text{(46)} \qquad R_{kij} = \frac{x_j^{(k)} y_i^{(k)}}{\displaystyle\sum_{\rho=1}^{n} \sum_{\sigma=1}^{n} b_{\rho\sigma} x_\sigma^{(k)} y_\rho^{(k)}}\,.$$

From this result we have for the kth term in the expansion of z_t in terms of $x_t^{(k)}$ by (16) and (17) the value

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji} z_i y_j^{(k)}}{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji} x_i^{(k)} y_j^{(k)}}\, x_t^{(k)} = \sum_{i=1}^{n}\sum_{j=1}^{n} R_{kjt} b_{ji} z_i = \frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_k}\sum_{i=1}^{n}\sum_{j=1}^{n} G_{jt}(\lambda) b_{ji} z_i d\lambda,$$

where Γ_k is a contour about $\lambda^{(k)}$ inclosing no other characteristic value of system (10). Then, if Γ is a contour inclosing all the characteristic values of systems (10), we have

$$(47)\qquad z_t = \frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}\sum_{i=1}^{n}\sum_{j=1}^{n} G_{jt}(\lambda) b_{ji} z_i d\lambda, \qquad t = 1, 2, \cdots, n.$$

Thus we replace the expansion for z_t by a contour integral. It is clear that any set of terms of the expansion may likewise be replaced by the same integral taken about a contour inclosing appropriate corresponding zeros of $\Delta(\lambda)$.

Contour integrals of this character were first employed by Birkhoff.* They served admirably in the convergence proofs associated with the expansions of functions in terms of functions defined by differential systems, these systems being of such character as to be realized as limiting cases of the algebraic systems here in consideration. We shall find of fundamental value in our general investigation various limiting forms of the contour integrals developed in this section.

It is possible (but not important for our purposes) to modify and extend the foregoing analysis so as to avoid the restriction that the zeros of $\Delta(\lambda)$ shall be distinct.

Let us now pass to the more general systems (1) and (2). For any given fixed value of h let Δ_h $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ denote the determinant in the first member of (3) and let Δ_{hij} $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ denote the cofactor of the element in the ith row and the jth column of this determinant. Then the Green's function G_{hij} of the hth system in (1) has the form

$$G_{hij} = \frac{\Delta_{hij}(\lambda_1, \lambda_2, \cdots, \lambda_r)}{\Delta_h(\lambda_1, \lambda_{,2} \cdots, \lambda_r)},$$

We assume for the present purpose that systems (1) [and also systems (2)] are so restricted that the solution of each for characteristic values of the λ's is unique (except for a constant factor) and that

$$\lim_{\lambda_1=\lambda_1^{(k)},\, \cdots,\, \lambda_r=\lambda_r^{(k)}} \frac{\Delta_h(\lambda_1, \lambda_2, \cdots, \lambda_r)}{\lambda_h - \lambda_h^{(k)}}$$

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* *Transactions of the American Mathematical Society,* 9 (1908): 373–395.

exists and is finite, where $\lambda_1^{(k)}, \cdots, \lambda_r^{(k)}$ is any set of characteristic values for systems (1) and (2). Then the function

$$G_{hij}(\lambda_1^{(i)}, \cdots, \lambda_{h-1}^{(k)}, \lambda_h, \lambda_{h+1}^{(k)}, \cdots, \lambda_r^{(k)})$$

of the complex variable λ_h has a pole of the first order at the point $\lambda_h = \lambda_h^{(k)}$. Let its residue at this point be denoted by R_{khij}.

In order to evaluate the residue R_{khij} we apply the earlier result of this section for R_{kij} to the system which results from the hth system in (1) on replacing each λ_i except λ_h by the corresponding characteristic value $\lambda_i^{(k)}$. Thus we have

$$R_{hkij} = \gamma_{kh} x_h^{(k)} y_i^{(k)}$$

where γ_{kh} is independent of i and j. Then we have

$$\prod_{h=1}^{r} R_{khi_hj_h} = \gamma_k \prod_{h=1}^{r} x_{hj_h}^{(k)} y_{hi_h}^{(k)},$$

where γ_k is independent of the i and the j.

Then if we write

$$G_{i_1j_1 \cdots i_rj_r}(\lambda_1, \lambda_2, \cdots, \lambda_r) = \prod_{h=1}^{r} G_{hi_hj_h}(\lambda_1, \lambda_2, \cdots, \lambda_r)$$

we have

$$\lim_{\lambda_1=\lambda_1^{(k)}, \dots, \lambda_r=\lambda_r^{(k)}} (\lambda_1 - \lambda_1^{(k)}) \cdots (\lambda_r - \lambda_r^{(k)}) G_{i_1j_1 \cdots i_rj_r}(\lambda_1, \cdots, \lambda_r) = \gamma_k \prod_{h=1}^{r} x_{hj_h}^{(k)} y_{hi_h}^{(k)};$$

whence it follows that

$$(48) \quad \lim_{\lambda_1=\lambda_1^{(k)}, \dots, \lambda_r=\lambda_r^{(k)}} (\lambda_1 - \lambda_1^{(k)}) \cdots (\lambda_r - \lambda_r^{(k)}) \times \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} G_{i_1t_1 \cdots i_rt_r} D_{j_1i_1 \cdots j_ri_r} \prod_{h=1}^{r} x_{kj_h}^{(h)}$$
$$= \gamma_k \prod_{h=1}^{r} x_{ht_h}^{(k)} \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{j_1i_1 \cdots j_ri_r} \prod_{h=1}^{r} x_{hj_h}^{(k)} y_{hi_h}^{(k)}.$$

Employing the hth system in (1) and proceeding by the method used earlier in this section for evaluating $x_t^{(k)}$ in terms of the Green's function, we have

$$x_{ht_h}^{(k)} = \sum_{s=1}^{r} (\lambda_s - \lambda_s^{(k)}) \sum_{i_h=1}^{n_h} \sum_{j_h=1}^{n_h} a_{shi_hj_h} x_{hj_h}^{(k)} G_{hi_ht_h}(\lambda_1, \cdots, \lambda_r), \qquad h = 1, 2, \cdots, r.$$

From this we may form the product

$$x_{1t_1}^{(k)} x_{2t_2}^{(k)} \cdots x_{rt_r}^{(k)}$$

and simplify its expression by taking the limit for any convenient approach of $\lambda_1, \cdots, \lambda_r$ to the characteristic values $\lambda_1^{(k)}, \cdots, \lambda_r^{(k)}$. Then, in a certain range of cases (if not always) it turns out that the value of the product in question is that of the first member of equation (48). *We confine attention to the cases in which this new condition is realized.* Then we have

$$1 = \gamma_k \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{j_1 i_1 \ldots j_r i_r} \prod_{h=1}^{r} x_{h j_h}^{(k)} y_{h i_h}^{(k)}.$$

Thence it follows that

$$\lim_{\substack{\lambda_1=\lambda_1^{(k)} \\ \cdot\ \cdot\ \cdot\ \cdot \\ \lambda_r=\lambda_r^{(k)}}} G_{j_1 i_1 \ldots j_r i_r}(\lambda_1, \cdots, \lambda_r) = \frac{\prod_{h=1}^{r} x_{h i_h}^{(k)} y_{h j_h}^{(k)}}{\sum_{\rho_1=1}^{n_1} \sum_{\sigma_1=1}^{n_1} \cdots \sum_{\rho_r=1}^{n_r} \sum_{\sigma_r=1}^{n_r} D_{\sigma_1 \rho_1 \ldots \sigma_r \rho_r} \prod_{h=1}^{r} x_{h \sigma_h}^{(k)} y_{h \rho_h}^{(k)}}.$$

From this result we have for the *k*th term of the expansion of $z_{t_1 t_2 \ldots t_r}$ afforded by (14) the value

$$\frac{1}{(2\pi\sqrt{-1})^r} \int_{\Gamma_{1k}} \int_{\Gamma_{2k}} \cdots \int_{\Gamma_{rk}} \left\{ \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} \sum_{j_r=1}^{n_r} D_{i_1 j_1 \ldots i_r j_r} z_{i_1 i_2 \ldots i_r} G_{j_1 t_1 \ldots j_r t_r} \right\} d\lambda_1 d\lambda_2 \cdots d\lambda_r,$$

where Γ_{hk} $(h = 1, 2, \cdots, r)$ is a contour in the λ_h-plane about the point $\lambda_h^{(k)}$ and containing in its interior no other characteristic value of λ_h. If we replace the contour Γ_{hk} $(h = 1, 2, \cdots, r)$ by Γ_h, a contour which includes within it all the characteristic values of λ_h, and perform the same multiple integration about such contours, we shall have the value of the function $z_{t_1 t_2 \ldots t_r}$. It is clear that we may form similarly the contour integral for any given partial sum of the series for $z_{t_1 t_2 \ldots t_r}$ afforded by relation (14).

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